

Recent Results and Open Problems on the Hydrodynamics of Disordered Asymmetric Exclusion and Zero-Range Processes

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Summary. This paper summarizes results and some open problems about the large-scale and long-time behavior of asymmetric, disordered exclusion and zero-range processes. These processes have randomly chosen jump rates at the sites of the underlying lattice \mathbf{Z}^d . The interesting feature is that for suitably distributed random rates there is a phase transition where the process behaves differently at high and low densities. Some of this distinction is visible on the hydrodynamic scale. But to fully understand the phase transition, results on a finer scale are needed.

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1. INTRODUCTION

This paper introduces some recent results on the hydrodynamics of disordered, asymmetric simple exclusion processes (SEP) and zero-range processes (ZRP). The *disorder* refers to the rates of jumping attached to the sites of the underlying lattice: The particles move on \mathbf{Z}^d , and each site $x \in \mathbf{Z}^d$ has a random variable α_x that influences the exponential rate at which particles leave site x . In SEP α_x is exactly the rate of jumping from x , and in ZRP α_x multiplies the rate $r(\eta(x))$ that depends on the number $\eta(x)$ of particles currently occupying site x . The *asymmetry* pertains to the jump probabilities $p(x, y)$, according to which a particle jumping from x chooses its new location y . We assume throughout that the kernel $p(x, y)$ is translation invariant so that $p(x, y) = p(0, y - x) \equiv p(y - x)$. Asymmetric jumping means that there typically is a drift: $\gamma \equiv \sum x p(x) \neq 0$. The assumption $\gamma \neq 0$ is not always necessary, but without it the limiting macroscopic conservation law becomes trivial. Some theorems require a stronger assumption of *total asymmetry*: the dimension $d = 1$, and all jumps proceed to the right: $p(1) = 1$.

The disorder can also be attached to particles, so that individual particles carry their own randomly chosen jump rates. We do not explicitly consider such processes. One special case, the totally asymmetric simple exclusion process (TASEP) with particlewise disorder, is partially covered by our discussion. This is because the gaps between the exclusion particles with random rates can be regarded as the occupation numbers of a ZRP with random rates on the sites. This special case has been studied in the physics literature as a model for traffic. See Krug (1998), Krug and Ferrari (1996), and their references.

The interesting phenomenon that appears in disordered particle systems is a phase transition where the process behaves differently at high and low densities. It occurs when the distribution of the random rate has a sufficiently thin tail at its left endpoint $c > 0$. Not much rigorous mathematical work exists on this phase transition.

A brief overview of the paper: In Section 2 the disordered ZRP is described, together with two theorems. In Section 3 the same is done for the disordered SEP. Section 4 lists four open problems. Section 5 contains some proofs and some comments on proofs. In particular, we included in Section 5 a rigorous construction of a disordered ZRP in \mathbf{Z}^d , and a proof of the invariance of a certain family of product measures. The construction is based on the percolation approach of Harris (1972).

The hope is that this paper would be at least partially accessible to the non-expert. This is the motivation for inclusion of the proofs in Section 5, which are often referred to but less often spelled out in the literature. For the same reason an attempt has been made to employ precise and complete notation. This may make the text somewhat heavy to follow at times, but the alternative is to risk confusing the reader who is not well-acquainted with disordered particle systems. Of course,

such an outcome may be unavoidable in any case.

Some familiarity with the subject of interacting particle systems is required for reading this paper. General references on particle systems are Durrett (1988, 1995), Griffeath (1979), and Liggett (1985). References on hydrodynamical limits are lectures by De Masi and Presutti (1991), the monograph of Spohn (1991), review papers by Ferrari (1994, 1996), and the soon-to-appear monograph of Kipnis and Landim.

Here are some references that are closely related, but not directly on the topic of the paper: Hydrodynamic limits for asymmetric processes with inhomogeneous but not random rates have been proved by Landim (1996), Covert and Rezakhanlou (1997), and Bahadoran (1998). Koukkous (1996) and Gielis et al. (1998) have studied the symmetric ZRP with random rates. [In the symmetric case the jump probabilities satisfy $p(x) = p(-x)$.]

Notational remarks. $\mathbf{Z}_+ = \{0, 1, 2, 3, \dots\}$. \mathbf{I}_A and $\mathbf{I}\{A\}$ denote the indicator random variable of the event A . δ_y is a delta function or a point mass at y , depending on the context: $\delta_y(x) = \mathbf{I}\{x = y\}$ for points x , and $\delta_x(A) = \mathbf{I}_A(x)$ for sets A .

2. THE DISORDERED ASYMMETRIC ZERO-RANGE PROCESS

First we describe a disordered ZRP on \mathbf{Z}^d with bounded, monotone jump rates. Let $\{p(x) : x \in \mathbf{Z}^d\}$ be a finite-range probability distribution, in other words $p(x) \geq 0$, $\sum p(x) = 1$, and for some fixed finite set $\mathcal{N} \subseteq \mathbf{Z}^d$, $p(x) = 0$ for $x \notin \mathcal{N}$. The rate of jumping from a site depends on the number of particles present through a function $r : \mathbf{Z}_+ \rightarrow [0, \infty)$, about which we assume that

$$(2.1) \quad 0 = r(0) < r(1) \leq r(2) \leq r(3) \leq r(4) \leq \dots$$

and

$$(2.2) \quad r(\infty) = \lim_{k \rightarrow \infty} r(k) < \infty.$$

The disorder comes in the form of random deceleration factors $\alpha_x \leq 1$ that depend on the sites x . Once $\alpha = (\alpha_x : x \in \mathbf{Z}^d)$ is picked, the dynamics operates as follows: If there are $\eta(x) \geq 1$ particles at site x , then at exponential rate $\alpha_x r(\eta(x))$ a single particle jumps away from site x . The new location of this particle is y with probability $p(y - x)$. This happens at each site x independently of what happens at other sites.

For fixed rates α , the generator of the process is

$$(2.3) \quad L^\alpha f(\eta) = \sum_{x, y \in \mathbf{Z}^d} p(y - x) \alpha_x r(\eta(x)) [f(\eta^{x, y}) - f(\eta)].$$

Here $\eta = (\eta(x) : x \in \mathbf{Z}^d)$ is an element of the state space $\mathcal{S} = \mathbf{Z}_+^{\mathbf{Z}^d}$ of the process, and $\eta^{x,y}$ is the configuration that results from the jump of a single particle from site x to site y : $\eta^{x,y} = \eta + \delta_y - \delta_x$. Section 5 contains a construction of this process, based on a percolation argument of Harris (1972). Due to assumption (2.2) the process can be started from any configuration $\eta \in \mathcal{S}$. We denote the process by $\eta(t) = (\eta(x, t) : x \in \mathbf{Z}^d)$, where $t \geq 0$ is the time variable.

The standing assumption is that α is an ergodic $[c, 1]$ -valued process for some constant $c \in (0, 1]$. Let Q denote the distribution of the process α on the space $\mathcal{A} = [c, 1]^{\mathbf{Z}^d}$. Fix c to be the left endpoint of the marginal distribution of α_0 , so that c is the largest number such that the process α is $[c, 1]$ -valued.

What makes the disordered ZRP tractable are invariant distributions that can be explicitly described. This description uses the same ideas as the process without disorder. If $\alpha_x \equiv 1$ (no disorder), among the extremal invariant distributions are the product measures μ_ψ on \mathcal{S} , indexed by a parameter $\psi \in [0, r(\infty))$, with marginals

$$(2.4) \quad \mu_\psi(\eta(x) = k) = Z(\psi)^{-1} \frac{\psi^k}{r(1) \cdots r(k)}, \quad x \in \mathbf{Z}^d, k \in \mathbf{Z}_+.$$

[See Andjel (1982).] For $k = 0$ the product in the denominator is interpreted as 1. $Z(\psi)$ is the normalization factor, defined by

$$(2.5) \quad Z(\psi) = \sum_{k=0}^{\infty} \frac{\psi^k}{r(1) \cdots r(k)}.$$

For the disordered process, fix a choice α for the rates. For real numbers $\varphi \in [0, r(\infty)c)$, let ν_φ^α denote the product probability measure on \mathcal{S} whose marginals vary from site to site, as given by

$$(2.6) \quad \nu_\varphi^\alpha(\eta(x) = k) = Z(\varphi/\alpha_x)^{-1} \frac{(\varphi/\alpha_x)^k}{r(1) \cdots r(k)}, \quad x \in \mathbf{Z}^d, k \in \mathbf{Z}_+.$$

Proposition 1. *For each choice of rates $\alpha \in \mathcal{A}$ and each value of $\varphi \in [0, r(\infty)c)$, the probability distribution ν_φ^α is invariant for the process with generator L^α .*

The *phase transition* of the disordered ZRP is the following situation: If Q is such that very slow sites are sufficiently rare, then the family of invariant distributions $\{\nu_\varphi^\alpha : \varphi \in [0, r(\infty)c)\}$ does not cover the entire range of densities $0 \leq \rho < \infty$. Instead, there is a critical density $\rho^* < \infty$ such that the equilibria ν_φ^α only exist for densities $\rho \in [0, \rho^*]$. To see this, set first

$$(2.7) \quad M(\psi) = \frac{1}{Z(\psi)} \sum_{k=0}^{\infty} \frac{k\psi^k}{r(1) \cdots r(k)}, \quad \psi \in [0, r(\infty)).$$

$M(\psi)$ is the density under μ_ψ . It is a strictly increasing function from $[0, r(\infty))$ onto $[0, \infty)$ and has an inverse function M^{-1} which we need to refer to below. For the disordered model the density ρ as a function of the parameter φ is defined by averaging over the random rates:

$$(2.8) \quad \rho(\varphi) = E^Q \left[\frac{1}{Z(\varphi/\alpha_0)} \sum_{k=0}^{\infty} \frac{k(\varphi/\alpha_0)^k}{r(1) \cdots r(k)} \right] = E^Q [M(\varphi/\alpha_0)].$$

Here E^Q denotes expectation over the distribution Q of α , and the random variable inside the expectation is α_0 . For a fixed equilibrium ν_φ^α the density $\rho(\varphi)$ can be realized as a spatial average

$$\rho(\varphi) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \eta(x) \quad \nu_\varphi^\alpha\text{-a.s., for } Q\text{-a.e. } \alpha.$$

By letting φ increase to its upper bound $r(\infty)c$, (2.8) shows that the maximal density is

$$(2.9) \quad \rho^* = E^Q [M(r(\infty)c/\alpha_0)].$$

This quantity may or may not be infinite, depending on the distribution Q . From (2.7) $M(r(\infty)) = \lim_{\psi \nearrow r(\infty)} M(\psi) = \infty$, so in particular if $Q(\alpha_0 = c) > 0$, then $\rho^* = \infty$. The interesting case with phase transition is the one where $Q(d\alpha_0)$ has a sufficiently thin tail as $\alpha_0 \searrow c$, to make the integral in (2.9) finite.

The function $\rho : [0, r(\infty)c) \rightarrow [0, \rho^*)$ is strictly increasing. Let $f : [0, \rho^*) \rightarrow [0, r(\infty)c)$ denote its inverse function. In other words, for $\rho \in [0, \rho^*)$, $f(\rho)$ is implicitly defined by

$$(2.10) \quad \rho = E^Q [M(f(\rho)/\alpha_0)].$$

Now we state a hydrodynamic limit for the disordered ZRP, due to Benjamini, Ferrari, and Landim (1996). For each choice of rates α , there is a sequence of zero-range processes indexed by n , generated by L^α . P_n^α denotes the probability measure on the probability space of the n th process $\{\eta_n(x, t) : x \in \mathbf{Z}^d, t \geq 0\}$, $n = 1, 2, 3, \dots$. The theorem is a weak law of large numbers for the empirical measure defined by

$$(2.11) \quad \pi_n(t) = n^{-d} \sum_{x \in \mathbf{Z}^d} \eta_n(x, t) \delta_{x/n},$$

where δ_x is a unit mass at the point $x \in \mathbf{R}^d$. The assumptions are the following:

(A.1) The transition probability $p(x)$ satisfies this irreducibility condition: for each $x, y \in \mathbf{Z}^d$ there exists a finite sequence $x = x_0, \dots, x_k = y$ such that $p(x_{i+1} - x_i) + p(x_i - x_{i+1}) > 0$ for all i .

(A.2) There exists a bounded continuous function u_0 on \mathbf{R}^d such that $\|u_0\|_\infty \leq M(r(\infty)\theta)$ for some $\theta < c$, and for each α , the initial distribution of the process η_n is given by

$$P_n^\alpha(\eta_n(x, 0) = k) = \mu_{M^{-1}(u_0(x/n))}(\eta(x) = k).$$

[Recall the definitions of μ_ψ and $M(\psi)$ from (2.4) and (2.7).]

(A.3) The marginal distribution of α_0 is supported by a finite set: For some $c = c_1 < c_2 < \dots < c_m \leq 1$, $Q(\alpha_0 \in \{c_1, \dots, c_m\}) = 1$.

Assumption (A.2) ensures that for some fixed $\varphi \in [0, r(\infty)c)$ and all α , all the initial distributions are stochastically dominated by ν_φ^α . This is true because, on the $\eta(x)$ -marginal $\nu_\varphi^\alpha = \mu_{\varphi/\alpha_x}$, and this dominates $\mu_{r(\infty)\theta}$ as long as $\varphi/\alpha_x \geq r(\infty)\theta$, which in turn is true for all $\alpha_x \in [c, 1]$ if $\varphi \geq r(\infty)\theta$.

Let $\gamma \in \mathbf{R}^d$ be the mean drift under $p(x)$: $\gamma = \sum_{x \in \mathbf{Z}^d} xp(x)$. Let $u(x, t)$ on $\mathbf{R}^d \times [0, \infty)$ be the unique entropy solution of the scalar conservation law

$$(2.12) \quad \partial u / \partial t + \gamma \cdot \nabla_x [f(u)] = 0, \quad u(x, 0) = u_0(x).$$

Let $C_0(\mathbf{R}^d)$ denote the space of compactly supported continuous functions on \mathbf{R}^d . This theorem was proved by Benjamini et al. (1996):

Theorem 1. *Under assumptions (A.1)–(A.3), the following holds for Q -a.e. α : For each $t > 0$, $\phi \in C_0(\mathbf{R}^d)$, and $\varepsilon > 0$:*

$$(2.13) \quad \lim_{n \rightarrow \infty} P_n^\alpha \left(\left| \pi_n(nt, \phi) - \int_{\mathbf{R}^d} \phi(x) u(x, t) dx \right| \geq \varepsilon \right) = 0.$$

The integral against $\pi_n(t)$ is defined by $\pi_n(t, \phi) = n^{-d} \sum_x \eta_n(x, t) \phi(x/n)$. The statement is that, in the topology of Radon measures on \mathbf{R}^d , $\pi_n(nt)$ converges to $u(x, t)dx$ in probability as $n \rightarrow \infty$.

The shortcoming of this result is that it does not indicate what happens on the hydrodynamic scale if the process starts at density above critical, that is, $u_0(x) > \rho^*$ for some or all x . In fact, assumption (A.3) makes $\rho^* = \infty$, so there can be no phase transition under these hypotheses.

Next we state a theorem that covers the hydrodynamics also in the high-density regime $\rho > \rho^*$ and admits more general initial distributions for the process. However, we pay a serious price for this strengthening: The theorem is valid only for

the most basic type of ZRP with rate function $r(k) = \mathbf{I}\{k \geq 1\}$. Furthermore, we are restricted to totally asymmetric jumps in one dimension: $d = 1$ and $p(1) = 1$, so jumps happen only to the right on \mathbf{Z} . Finally, we assume that the process of rates $(\alpha_x : x \in \mathbf{Z})$ is i.i.d.

In this case the measures ν_φ^α are products of geometric distributions:

$$(2.14) \quad \nu_\varphi^\alpha(\eta(x) = k) = (1 - \varphi/\alpha_x)(\varphi/\alpha_x)^k, \quad x \in \mathbf{Z}, k \in \mathbf{Z}_+.$$

Now $M(\psi) = \psi/(1 - \psi)$, so the definition of the critical density becomes

$$\rho^* = c \int_{[c,1]} (\alpha_0 - c)^{-1} Q(d\alpha_0).$$

From this formula it is plainly obvious how the tail of $Q(d\alpha_0)$ at $\alpha_0 = c+$ determines whether $\rho^* < \infty$ or not, that is, whether phase transition happens or not.

For $\rho \in [0, \rho^*)$ the flux function $f(\rho)$ is defined implicitly by the equation

$$(2.15) \quad \rho = f(\rho) \int_{[c,1]} (\alpha_0 - f(\rho))^{-1} Q(d\alpha_0).$$

If $\rho^* < \infty$, set

$$(2.16) \quad f(\rho) = c \text{ for } \rho \geq \rho^*.$$

This makes f a nondecreasing and concave function on $[0, \infty)$.

Again we assume we have a sequence of processes $\eta_n(t)$ and corresponding probability measures P_n^α . The assumption on initial distributions is this:

(A.4) Suppose $u_0(x)$ is a nonnegative locally integrable function on \mathbf{R} . Assume that this holds for Q -a.e. α : For all $\phi \in C_0(\mathbf{R})$ and $\varepsilon > 0$,

$$(2.17) \quad \lim_{n \rightarrow \infty} P_n^\alpha \left(\left| \pi_n(0, \phi) - \int_{\mathbf{R}} \phi(x) u_0(x) dx \right| \geq \varepsilon \right) = 0.$$

Previously this assumption was a consequence of assumption (A.1) so it was not stated explicitly. Let $u(x, t)$ on $\mathbf{R} \times [0, \infty)$ be the unique entropy solution of

$$(2.18) \quad \partial u / \partial t + \partial f(u) / \partial x = 0, \quad u(x, 0) = u_0(x)$$

where f is the function defined by (2.15)–(2.16). Then we have a theorem due to Seppäläinen and Krug (1998):

Theorem 2. Assume that $d = 1$, $p(1) = 1$, $r(k) = \mathbf{I}\{k \geq 1\}$, and that Q is an i.i.d. distribution for $\alpha = (\alpha_x)$. Then under assumption (A.4) the following holds for Q -a.e. α : For each $t > 0$, $\phi \in C_0(\mathbf{R})$, and $\varepsilon > 0$:

$$(2.19) \quad \lim_{n \rightarrow \infty} P_n^\alpha \left(\left| \pi_n(nt, \phi) - \int_{\mathbf{R}} \phi(x) u(x, t) dx \right| \geq \varepsilon \right) = 0.$$

In the phase transition case this theorem has an interesting consequence: Suppose the initial profile satisfies $u_0(x) \geq \rho^*$ everywhere on \mathbf{R} . Then since f is constant for this range of densities, it follows that $u(x, t) = u_0(x)$ for all $t > 0$. In other words, the profile does not change on the hydrodynamic scale.

3. THE DISORDERED ASYMMETRIC EXCLUSION PROCESS

The disordered SEP is less well understood than the ZRP. The reason is that no invariant distributions have been found. Presently we can prove the existence of a hydrodynamic limit for the totally asymmetric SEP, in dimension one. But the flux function f of the macroscopic equation (2.18) remains unknown. The theorem covers a more general totally asymmetric SEP, namely the so-called K -exclusion, where each site admits K particles instead of just one. The state space is $\mathcal{S} = \{0, \dots, K\}^{\mathbf{Z}}$, and we write again $\eta = (\eta(x) : x \in \mathbf{Z}) \in \mathcal{S}$ for the particle configurations. When the rates α have been chosen, the generator is

$$(3.1) \quad L^\alpha f(\eta) = \sum_{x \in \mathbf{Z}} \alpha_x \mathbf{I}_{\{\eta(x) \geq 1, \eta(x+1) \leq K-1\}} [f(\eta^{x, x+1}) - f(\eta)].$$

In other words, a jump occurs from site x to $x + 1$ at rate α_x , provided site x is not empty and site $x + 1$ has less than K particles. Consider K fixed but arbitrary. As before, assume we have a sequence of totally asymmetric K -exclusion processes $\eta_n(t)$, with probability measures P_n^α when the rates α are fixed. As for Theorem 2, we only assume that the initial distributions of the processes have a well-defined macroscopic profile:

(A.5) Suppose u_0 is a bounded measurable function on \mathbf{R} such that $0 \leq u_0(x) \leq K$. Assume (2.17) holds for all $\phi \in C_0(\mathbf{R})$ and $\varepsilon > 0$, for Q -a.e. α .

The theorem about the existence of the hydrodynamic limit is proved in Seppäläinen (1998):

Theorem 3. Fix a positive integer K and an i.i.d. distribution Q for $\alpha = (\alpha_x)$. Then there exists a concave function f_K on $[0, K]$ that depends on Q such that, if $u(x, t)$ is the unique entropy solution of (2.18) with flux $f = f_K$, then under

assumption (A.5), the limit in (2.19) holds for Q -a.e. α , for each $t > 0$, $\phi \in C_0(\mathbf{R})$, and $\varepsilon > 0$.

The entropy solution to (2.18) with $f = f_K$ and u_0 bounded measurable can be constructed without explicit reference to the p.d.e: Pick a function U_0 such that $U'_0 = u_0$ a.e. on \mathbf{R} . Let f_K^* be the concave conjugate of f_K :

$$f_K^*(x) = \inf_{0 \leq \rho \leq K} \{x\rho - f_K(\rho)\}.$$

Set

$$(3.2) \quad U(x, t) = \sup_{y \in \mathbf{R}} \{U_0(y) + tf_K^*((x - y)/t)\}.$$

Finally, let $u(x, t) = (\partial/\partial x)U(x, t)$, a derivative that is defined at least a.e. Lax (1957) discusses this construction of the entropy solution of a scalar conservation law. See also Section 3.4 in Evans (1998). The conjugate f_K^* has a probabilistic meaning in this context: It is the macroscopic shape of a growth model associated to the K -exclusion process [Seppäläinen (1998)].

4. OPEN PROBLEMS

Problem 1. Extend Theorem 1 to describe hydrodynamic behavior also at high density $\rho > \rho^*$. Or, equivalently, extend Theorem 2 to more general ZRP's. The proof of Theorem 2 in Seppäläinen and Krug (1998) depends on special properties of the totally asymmetric ZRP with rate function $r(k) = \mathbf{I}\{k \geq 1\}$. The proof of Benjamini et al. for Theorem 1 follows a strategy invented by Rezakhanlou (1991), and may be a better candidate for generalization.

Problem 2. In the phase transition case, the hydrodynamic theorem reveals only trivial behavior at high density $\rho > \rho^*$ (the profile does not move, see Theorem 2). It is expected that on a finer space scale one can see inhomogeneities develop, specifically, that the particles concentrate on exceptionally slow sites. No rigorous results exist to describe these phenomena. Krug (1998) and Seppäläinen and Krug (1998) discuss this in terms of the exclusion version of the totally asymmetric ZRP.

Problem 3. An interesting open problem for the disordered ZRP concerns the weak convergence of the process: Fix the rates α , and take the initial distribution of $\eta(0)$ spatially ergodic with density ρ . Does the distribution of $\eta(t)$ converge weakly, as $t \rightarrow \infty$, to one of the equilibria ν_φ^α ? If $\rho > \rho^*$ in the phase transition case, does $\eta(t)$ converge weakly to the equilibrium with density ρ^* ? For the standard ZRP results of this type were proved by Andjel et al. (1985).

Problem 4. For the disordered TASEP, any new rigorous results would be welcome. For example, is there a phase transition similar to the one for disordered ZRP? Does the flux function $f_K(\rho)$ have a flat segment on an interval around $\rho = K/2$?

5. COMMENTS ON THE PROOFS

5.1 Construction and invariant distributions. To construct the disordered ZRP with generator (2.3), start by giving each site $x \in \mathbf{Z}^d$ an independent rate $r(\infty)$ Poisson point process \mathcal{T}^x on the time line $(0, \infty)$. [Here we make use of the assumption $r(\infty) < \infty$.] To the i th epoch of \mathcal{T}^x attach two random objects: A random threshold U_i^x uniformly distributed on $[0, r(\infty)]$ and independent of everything else, and a destination site y_i^x chosen with probability $p(y_i^x - x)$, again independently of everything else. Fix the rates $\alpha = (\alpha_x : x \in \mathbf{Z}^d)$ and an initial configuration $\eta = (\eta(x) : x \in \mathbf{Z}^d)$. Informally speaking, from these ingredients the construction of the dynamics goes as follows: Suppose τ is an epoch of \mathcal{T}^x with threshold U and destination site y , and the dynamics $\eta(t)$ has been constructed for times $0 \leq t < \tau$. If $U \geq \alpha_x r(\eta(\tau-))$, do nothing. If $U < \alpha_x r(\eta(\tau-))$, move one particle from site x to site y ; in other words, set

$$\begin{aligned} \eta(x, \tau) &= \eta(x, \tau-) - 1, \\ \eta(y, \tau) &= \eta(y, \tau-) + 1, \\ \text{and } \eta(z, \tau) &= \eta(z, \tau-) \quad \text{for } z \neq x, y. \end{aligned}$$

Subsequently site x lies dormant until the next epoch of \mathcal{T}^x . However, the lattice \mathbf{Z}^d is infinite, so this induction never even begins because there is no first epoch among the point processes $\{\mathcal{T}^x : x \in \mathbf{Z}^d\}$.

To make the construction rigorous, we show that there exists a fixed time $t_0 > 0$ such that, starting with an arbitrary η in the state space \mathcal{S} , the evolution $\eta(t)$ can be computed for $t \in [0, t_0]$. Since t_0 is independent of η , the construction can be repeated, starting with $\eta(t_0)$, and extended to $[0, 2t_0]$. And so on, to arbitrarily large times.

Given a fixed number $t_0 > 0$ and the Poisson processes $\{\mathcal{T}^x\}$, construct the following random graph, with vertex set \mathbf{Z}^d : Connect x and y with an edge if $x - y$ or $y - x$ lies in \mathcal{N} , and either \mathcal{T}^x or \mathcal{T}^y has an epoch in $[0, t_0]$. Recall that $p(z) = 0$ for z outside \mathcal{N} .

Lemma 5.1. *For small enough $t_0 > 0$, the random graph thus constructed has no infinite connected components, for almost every realization of $\{\mathcal{T}^x\}$.*

Before proving the lemma, let us observe how it solves the construction problem: All the sites y that could possibly influence the evolution at site x up to time t_0 are connected to x in the random graph. Since x lies in a finite connected component \mathcal{C} , the point process $\cup_{x \in \mathcal{C}} \mathcal{T}^x$ has only finitely many epochs in $[0, t_0]$. Consequently the evolution $\eta(z, t)$ can be computed for $z \in \mathcal{C}$ and $t \in [0, t_0]$ from the informal rule spelled out above, by considering the finitely many epochs in their temporal order. This procedure is repeated for all connected components.

Proof of Lemma 5.1. By translation invariance, it suffices to show that the origin is almost surely connected to only finitely many vertices. Let $\mathcal{N}^* = \mathcal{N} \cup (-\mathcal{N})$ and $K = |\mathcal{N}^*|$, the number of vertices x such that either x or $-x$ lies in \mathcal{N} . Call x_0, x_1, \dots, x_n a self-avoiding path with n edges in the random graph if $x_i \neq x_j$ for $i \neq j$ and there is an edge between x_i and x_{i+1} for each i . In particular, then $x_{i+1} - x_i \in \mathcal{N}^*$ for each i .

Let $|\cdot|$ be any norm on \mathbf{R}^d , and $R = \max\{|x| : x \in \mathcal{N}^*\}$. If the origin is connected to a vertex x with $|x| > L$, there is a self-avoiding path with at least L/R edges starting at the origin. The probability that a self-avoiding path with $2n-1$ edges starts at the origin is at most

$$K^{2n-1} (1 - e^{-2r(\infty)t_0})^n.$$

To see this, note first that the factor K^{2n-1} is an upper bound on the number of such paths. If $0 = x_0, x_1, \dots, x_{2n-1}$ is such a path, the n edges $(x_0, x_1), (x_2, x_3), \dots, (x_{2n-2}, x_{2n-1})$ are present independently of each other, and each with probability $1 - e^{-2r(\infty)t_0}$ [at least one of $\mathcal{T}^{x_{2i}}$ and $\mathcal{T}^{x_{2i+1}}$ must have an epoch in $[0, t_0]$, and each \mathcal{T}^{x_i} has rate $r(\infty)$]. Pick t_0 small enough so that $K^2(1 - e^{-2r(\infty)t_0}) < 1$. Then by Borel-Cantelli, self-avoiding paths from the origin have a finite upper bound on their length, almost surely. \square

This approach to the construction of a particle system is due to Harris (1972). Our presentation followed Durrett (1995).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space whose sample point ω represents a realization of the Poisson processes $\{\mathcal{T}^x\}$, the random thresholds $\{U_i^x\}$, and the destination sites $\{y_i^x\}$. We constructed the random path $\eta(\cdot) = \{\eta(x, t) : x \in \mathbf{Z}^d, t \geq 0\}$ as a function of the initial state η , the rates α , and a sample point ω . Since the Poisson processes are Markovian, and the random choices of threshold U and destination state y are independent of everything, the process $\eta(\cdot)$ is a time-homogeneous Markov process. Let $\mathcal{D} = \mathcal{D}([0, \infty), \mathcal{S})$ denote the space of right-continuous \mathcal{S} -valued functions on $[0, \infty)$ that have left limits at each point $t \in [0, \infty)$. For fixed (α, η) the path $\eta(\cdot)$ is a \mathcal{D} -valued random variable, and it has a probability distribution $P^{\alpha, \eta}$ on \mathcal{D} induced from the measure \mathbb{P} on Ω . Write $E^{\alpha, \eta}$ for the expectation under $P^{\alpha, \eta}$.

Next we prove Proposition 1 about the invariant distributions. For this we restrict the process to a cube $\Lambda_k = \{-k, \dots, k\}^d \subseteq \mathbf{Z}^d$. The jump probabilities $p(y - x)$ are then interpreted with periodic boundary conditions, and become

$$(5.1) \quad p_k(x, y) = \sum \{p(z - x) : z = y + (2k + 1)w \text{ for some } w \in \mathbf{Z}^d\}$$

for $x, y \in \Lambda_k$. The finite-volume generator is

$$(5.2) \quad L_k^\alpha f(\eta) = \sum_{x, y \in \Lambda_k} p_k(x, y) \alpha_x r(\eta(x)) [f(\eta^{x, y}) - f(\eta)].$$

L_k^α generates a Markov jump process with uniformly bounded rates on the countable state space $\mathcal{S}_k = \mathbf{Z}_+^{\Lambda_k}$. Existence of this process follows from standard textbook material. Write $E_k^{\alpha, \eta}$ for expectations under distributions of the process restricted to Λ_k . Notice that if f is a cylinder function on \mathcal{S} , $L_k^\alpha f = L^\alpha f$ for all large enough k .

Define dual transition probabilities by $p_k^*(x, y) = p_k(y, x)$ and $p^*(x) = p(-x)$. Then p_k^* is obtained from p^* exactly as p_k from p according to (5.1). Let $L_k^{\alpha, *}$ and $L^{\alpha, *}$ be the generators with kernels p_k^* and p^* in place of p_k and p .

Lemma 5.2. *Let $\alpha \in \mathcal{A}$ and $\varphi \in [0, r(\infty)c)$, and let ν_φ^α be the product probability measure defined by (2.14). Then for all bounded measurable functions f, g on \mathcal{S}_k ,*

$$\nu_\varphi^\alpha(gL_k^\alpha f) = \nu_\varphi^\alpha(fL_k^{\alpha, *}g).$$

Proof. Start by writing

$$\begin{aligned} \nu_\varphi^\alpha(gL_k^\alpha f) &= \sum_{x, y \in \Lambda_k} p_k(x, y) \alpha_x \nu_\varphi^\alpha[r(\eta(x))g(\eta)f(\eta^{x, y})] \\ &\quad - \sum_{x, y \in \Lambda_k} p_k(x, y) \alpha_x \nu_\varphi^\alpha[r(\eta(x))g(\eta)f(\eta)] \\ &\equiv A_1 - A_2, \end{aligned}$$

where the last equality defines the quantities A_1 and A_2 .

Continue first with A_1 : For any x, y let $\tilde{\nu}$ denote the marginal distribution of $\tilde{\eta} = (\eta(z) : z \neq x, y)$. Abbreviate $R(m) = r(1) \cdots r(m)$. Then for a fixed x , the sum over y in A_1 can be expressed as

$$\begin{aligned} &\sum_{y \in \Lambda_k} p_k(x, y) \alpha_x \nu_\varphi^\alpha[r(\eta(x))g(\eta)f(\eta^{x, y})] \\ &= \sum_{y \in \Lambda_k} p_k(x, y) \alpha_x \int \tilde{\nu}(d\tilde{\eta}) \sum_{\substack{m_x \geq 1 \\ m_y \geq 0}} Z(\varphi/\alpha_x)^{-1} Z(\varphi/\alpha_y)^{-1} R(m_x)^{-1} R(m_y)^{-1} \\ &\quad (\varphi/\alpha_x)^{m_x} (\varphi/\alpha_y)^{m_y} r(m_x) g(\tilde{\eta}, m_x, m_y) f(\tilde{\eta}, m_x - 1, m_y + 1) \\ &= \sum_{y \in \Lambda_k} p_k(x, y) \alpha_y \int \tilde{\nu}(d\tilde{\eta}) \sum_{\substack{n_x \geq 0 \\ n_y \geq 1}} Z(\varphi/\alpha_x)^{-1} Z(\varphi/\alpha_y)^{-1} R(n_x)^{-1} R(n_y)^{-1} \\ &\quad (\varphi/\alpha_x)^{n_x} (\varphi/\alpha_y)^{n_y} r(n_y) g(\tilde{\eta}, n_x + 1, n_y - 1) f(\tilde{\eta}, n_x, n_y) \\ &= \sum_{y \in \Lambda_k} p_k(x, y) \alpha_y \nu_\varphi^\alpha[r(\eta(y))g(\eta^{y, x})f(\eta)]. \end{aligned}$$

In the above calculation we first used the definition (2.14) of ν_φ^α . The expectation is taken separately over the marginal distributions of $\tilde{\eta}$, $\eta(x)$, and $\eta(y)$. In the second expression, m_x and m_y are summation variables that represent integration over the distributions of $\eta(x)$ and $\eta(y)$. Because $r(m_x) = 0$ for $m_x = 0$, we sum over $m_x \geq 1$. In the second equality we do a change of variable in the sum: $n_x = m_x - 1$ and $n_y = m_y + 1$. The last equality is just the definition (2.14) of ν_φ^α again.

Now sum over $x \in \Lambda_k$ and interchange the summation indices x, y to get

$$A_1 = \sum_{x, y \in \Lambda_k} p_k^*(x, y) \alpha_x \nu_\varphi^\alpha [r(\eta(x)) g(\eta^{x, y}) f(\eta)].$$

In A_2 simply observe that $\sum_y p_k(x, y) = 1 = \sum_y p_k^*(x, y)$ and write

$$\begin{aligned} A_2 &= \sum_{x \in \Lambda_k} \left\{ \sum_{y \in \Lambda_k} p_k(x, y) \right\} \alpha_x \nu_\varphi^\alpha [r(\eta(x)) g(\eta) f(\eta)] \\ &= \sum_{x, y \in \Lambda_k} p_k^*(x, y) \alpha_x \nu_\varphi^\alpha [r(\eta(x)) g(\eta) f(\eta)]. \end{aligned}$$

Combining gives

$$\nu_\varphi^\alpha (g L_k^\alpha f) = A_1 - A_2 = \nu_\varphi^\alpha (f L_k^{\alpha, *}) g$$

and the lemma is proved. \square

Proof of Proposition 1. Let $t_0 > 0$ be the number chosen in Lemma 5.1. Since $\eta(t)$ is Markovian, invariance follows if we prove that, for any cylinder function f on \mathcal{S} ,

$$(5.3) \quad \int_{\mathcal{S}} E^{\alpha, \eta} [f(\eta(t))] \nu_\varphi^\alpha (d\eta) = \int_{\mathcal{S}} f d\nu_\varphi^\alpha$$

for all $t \in [0, t_0]$.

For any $\eta \in \mathcal{S}$, let η_{Λ_k} denote its restriction to Λ_k . Let $\eta^k(\cdot)$ denote the process in \mathcal{S}_k generated by L_k^α . (This of course is *not* the restriction of the process $\eta(\cdot)$ to Λ_k .) Taking $g \equiv 1$ in Lemma 5.2 shows that ν_φ^α is invariant for the process $\eta^k(\cdot)$, so we have

$$(5.4) \quad \int_{\mathcal{S}} E_k^{\alpha, \eta_{\Lambda_k}} [f(\eta^k(t))] \nu_\varphi^\alpha (d\eta) = \int_{\mathcal{S}} f d\nu_\varphi^\alpha$$

for all k large enough so that f can be regarded as a function on \mathcal{S}_k . [Here again we rely on basic facts of continuous-time Markov chains on countable state spaces. The whole point is of course that the space \mathcal{S} is not countable, so we need something more to pass from Lemma 5.2 to (5.3).] It remains to argue that

$$(5.5) \quad E_k^{\alpha, \eta_{\Lambda_k}} [f(\eta^k(t))] \rightarrow E^{\alpha, \eta} [f(\eta(t))]$$

boundedly, as $k \rightarrow \infty$, for any fixed $t \in [0, t_0]$ and $\eta \in \mathcal{S}$. For fixed (α, η) , we can construct the processes $\eta(\cdot)$ and $\eta^k(\cdot)$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We only need to interpret the destination sites y_i^x “modulo Λ_k ” for the process $\eta^k(\cdot)$. Then both $f(\eta^k(t))$ and $f(\eta(t))$ are functions on Ω , and both expectations in (5.5) are integrals over $(\Omega, \mathcal{F}, \mathbb{P})$. Since f is bounded, it suffices to show that

$$(5.6) \quad f(\eta^k(t)) \rightarrow f(\eta(t)) \quad \text{as } k \rightarrow \infty,$$

almost surely on $(\Omega, \mathcal{F}, \mathbb{P})$.

Pick k_0 large enough so that $f(\eta)$ is completely determined by $(\eta(x) : x \in \Lambda_{k_0})$. Let \mathcal{C}_{k_0} be the random set of vertices connected to Λ_{k_0} in the random graph discussed in Lemma 5.1. By that lemma, \mathcal{C}_{k_0} is almost surely finite. But then $f(\eta^k(t)) = f(\eta(t))$ as soon as k is large enough so that $\mathcal{C}_{k_0} \subseteq \Lambda_k$, because then all the transitions that affect the value of f are performed identically for $\eta(\cdot)$ and $\eta^k(\cdot)$ throughout the time interval $[0, t_0]$. This proves (5.6) and thereby (5.5), and then (5.4) turns into (5.3) as $k \rightarrow \infty$. \square

5.2 The hydrodynamic limits. Theorem 1 is Theorem 3.1 from Benjamini et al. (1996). It is proved by deriving a microscopic version of Kruzkov’s entropy inequalities that characterize the entropy solution of a conservation law. This idea for proving hydrodynamic limits of asymmetric processes is due to Rezakhanlou (1991).

Theorem 2 follows from Theorem 2 in Seppäläinen and Krug (1998). The discussion in this paper is formulated for a totally asymmetric exclusion process where the random rates are attached to the particles. To obtain results for the ZRP with site disorder, interpret the occupation numbers (η_i) as the gaps (= number of empty sites) between successive exclusion particles. The proof uses a special construction that works for the totally asymmetric ZRP with rate function $r(k) = \mathbf{I}\{k \geq 1\}$. Whether the technique can be somehow generalized to deduce results for other choices of $r(k)$ is presently unclear.

Theorem 3 is proved in Seppäläinen (1998). The approach is similar to the one in Seppäläinen and Krug (1998). It involves coupling the exclusion process with a countable collection of realizations of the same process but with a simple initial configuration. This coupling has the property that a microscopic version of the Lax-Oleinik formula (3.2) holds almost surely.

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